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$M2$-brane solutions in $AdS_7 \times S^4$

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Abstract: We consider different M2-brane configurations in the M-theory $AdS_7 \times S^4$ background, with field theory dual $A_{N-1}(2,0)$ SCFT. New membrane solutions are found and compared with the recently obtained ones.

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1. Introduction

The paper [1] by Gubser, Klebanov and Polyakov on the semi-classical limit of the AdS/CFT duality has inspired a lot of interest in the investigation of the existing connections between the classical string solutions, their semi-classically quantized versions and the relevant objects on the field theory side. Different string configurations have been considered, describing rotating, pulsating or orbiting strings. Much attention has been paid to string solutions in type IIB $AdS_5 \times S^5$ background with field theory dual $\mathcal{N} = 4$ SU(N) SYM in four dimensional flat space-time. Moreover, the string dynamics has been investigated in many other string theory backgrounds, known to have field theory dual descriptions in different dimensions, with different number of (or without) supersymmetries, conformal or non-conformal. Besides, membrane solutions in M-theory backgrounds have been obtained [2]–[5]. In [2]–[4], M2-brane configurations have been considered in $AdS_7 \times S^4$ space-time, with field theory dual $A_{N-1}(2,0)$ SCFT. Rotating membrane solution in AdS_7 have been obtained in [2]. Rotating and boosted membrane configuration was investigated in [3]. Multiwrapped circular membrane, pulsating in the radial direction of AdS_7, has been considered in [4]. The article [5] is devoted to the investigation of rotating membranes on G_2 manifolds.

Here, we will be interested in obtaining new membrane solutions in $AdS_7 \times S^4$ M-theory background. In section 2, we give brief description of the recently received M2-brane solutions in this space-time. In section 3, we firstly settle the framework, which we will work in.[1] Then, we proceed to find several M2-brane solutions, based on two different types of membrane embedding. The generic formulas, necessary for our calculations in this section, are collected in appendix.

[1] Actually, we will use the general approach developed in [6].

2. Short review of the recent M2-brane solutions in $AdS_7 \times S^4$

Let us review briefly some of the results obtained recently in [2]–[4], concerning the M2-brane dynamics on $AdS_7 \times S^4$ background.

In [2], a rotating membrane in AdS_7 was considered. The background metric is taken in global coordinates

$$ds^2_{AdS_7} = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left[d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\phi d\Omega_3^2\right)\right].$$

The M2-brane worldvolume coordinates are (τ, σ, φ) and the rotating membrane configuration is given by the ansatz

$$t = \tau, \qquad \rho = \rho(\sigma), \qquad \theta = \theta(\varphi), \qquad \phi = \omega\tau. \tag{2.1}$$

Therefore, the metric seen by the M2-brane is

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

This background does not depend on the coordinates t and ϕ, which leads to the conservation of the corresponding generalized momenta - the energy and the spin of the membrane. For the configuration (2.1), they were found to be

$$\begin{aligned}
E &= 4N\int_0^{\rho_0} d\rho \int_{\theta_1}^{\theta_2} d\theta \frac{\cosh^2\rho\sinh\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho\sin^2\theta}}, \\
s &= 4N\omega\int_0^{\rho_0} d\rho \int_{\theta_1}^{\theta_2} d\theta \frac{\sinh^3\rho\sin^2\theta}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho\sin^2\theta}}.
\end{aligned}$$

Rotating and boosted membrane configuration was investigated in [3]. The following coordinates for the $AdS_7 \times S^4$ metric have been used

$$\begin{aligned}
l_p^{-2} ds^2_{AdS_7\times S^4} &= 4R^2\Big\{-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\psi_1^2 + \cos^2\psi_1 d\psi_2^2 + \sin^2\psi_1 d\Omega_3^2\right) + \\
&\qquad + \frac{1}{4}\left[d\alpha^2 + \cos^2\alpha d\theta^2 + \sin^2\alpha\left(d\beta^2 + \cos^2\beta d\gamma^2\right)\right]\Big\}, \\
d\Omega_3^2 &= d\psi_3^2 + \cos^2\psi_3 d\psi_4^2 + \cos^2\psi_3\cos^2\psi_4 d\psi_5^2, \qquad R^3 = \pi N.
\end{aligned} \tag{2.2}$$

The coordinates which parameterize the membrane worldvolume are chosen to be (ξ_1, ξ_2, ξ_3) $= (\tau, \delta, \sigma)$. Then, the considered M2-brane embedding can be written as follows

$$t = \kappa\tau, \qquad \rho = \rho(\sigma), \qquad \psi_1 = \frac{\pi}{4}, \qquad \psi_2 = \sqrt{2}a\delta, \qquad \psi_5 = \sqrt{2}\omega\tau, \qquad \theta = 2\nu\tau, \tag{2.3}$$

and all other coordinates set to zero. Hence, the background felt by the membrane is

$$ds^2 = (2l_p R)^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \frac{1}{2}\sinh^2\rho\left(d\psi_2^2 + d\psi_5^2\right) + \frac{1}{4}d\theta^2\right].$$

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This metric does not depend on four coordinates - t, ψ_2, ψ_5 and θ. The conserved quantities, corresponding to the Killing vectors $\partial/\partial t$, $\partial/\partial\psi_5$ and $\partial/\partial\theta$, have been obtained to be given by the equalities

$$\begin{aligned}
E &= \frac{4R^3}{\pi}\kappa\int_0^{\rho_0}\frac{\sinh\rho\cosh^2\rho d\rho}{\sqrt{(\kappa^2-\nu^2)\cosh^2\rho-(\omega^2-\nu^2)\sinh^2\rho}}\,,\\
S &= \frac{4R^3}{\pi}\omega\int_0^{\rho_0}\frac{\sinh^3\rho d\rho}{\sqrt{(\kappa^2-\nu^2)\cosh^2\rho-(\omega^2-\nu^2)\sinh^2\rho}}\,,\\
J &= \frac{4R^3}{\pi}\nu\int_0^{\rho_0}\frac{\sinh\rho d\rho}{\sqrt{(\kappa^2-\nu^2)\cosh^2\rho-(\omega^2-\nu^2)\sinh^2\rho}}\,.
\end{aligned}$$

It was pointed out in [3] that there exists the following connection between the energy E, the spin S and the R-charge J of the membrane

$$E = \frac{\kappa}{\omega}S + \frac{\kappa}{\nu}J\,.$$

Then, this constraint has been used to determine the dependence of E on S and J.

Another type of M2-brane configuration - multiwrapped circular membrane pulsating in the radial direction of AdS_7, has been considered in [4]. The coordinates on $AdS_7\times S^4$ and on the M2-brane worldvolume are chosen as in [3]. The membrane embedding is given by the ansatz

$$t=\tau\,,\qquad \rho=\rho(\tau)\,,\qquad \psi_1=\frac{\pi}{4}\,,\qquad \psi_2=\sqrt{2}a\delta\,,\qquad \psi_5=\sqrt{2}m\sigma\,. \tag{2.4}$$

It follows from here that the metric seen by the M2-brane is

$$ds^2=(2l_pR)^2\left[-\cosh^2\rho dt^2+d\rho^2+\frac{1}{2}\sinh^2\rho\left(d\psi_2^2+d\psi_5^2\right)\right]. \tag{2.5}$$

The relevant action for such membrane configuration reads [4]:

$$I=-\left(2R\right)^3am\int dt\sinh^2\rho\sqrt{\cosh^2\rho-\dot{\rho}^2}\,. \tag{2.6}$$

3. New M2-brane solutions in $AdS_7\times S^4$

In considering the M2-brane dynamics, we will use the following action for a membrane moving in curved space-time with metric tensor $g_{MN}(x)$, and interacting with a background 3-form gauge field $b_{MNP}(x)$

$$\begin{aligned}
S &= \int d^3\xi\mathcal{L} \qquad\qquad (3.1)\\
&= \int d^3\xi\left\{\frac{1}{4\lambda^0}\left[g_{MN}\left(X\right)\left(\partial_0-\lambda^i\partial_i\right)X^M\left(\partial_0-\lambda^j\partial_j\right)X^N-\left(2\lambda^0T_2\right)^2\times\right.\right.\\
&\qquad\left.\left.\times\det\left(g_{MN}(X)\partial_iX^M\partial_jX^N\right)\right]+T_2b_{MNP}(X)\partial_0X^M\partial_1X^N\partial_2X^P\right\},\\
\xi &= (\xi^0,\xi^1,\xi^2)=(\tau,\delta,\sigma)\,,\qquad \partial_m=\frac{\partial}{\partial\xi^m}\,,\\
m &= (0,i)=(0,1,2),\qquad M=(0,1,\ldots,10)\,,
\end{aligned}$$

where λ^m are Lagrange multipliers, $x^M = X^M(\xi)$ are the membrane embedding coordinates, and T_2 is its tension. This action is classically equivalent to the Nambu-Goto type action[2]

$$S^{NG} = -T_2 \int d^3\xi \left[\sqrt{-\det\left(\partial_m X^M \partial_n X^N g_{MN}(X)\right)} - \frac{1}{6}\varepsilon^{mnp}\partial_m X^M \partial_n X^N \partial_p X^P b_{MNP}(X)\right], \tag{3.2}$$

and to the Polyakov type action

$$\begin{aligned} S^P = -\frac{T_2}{2}\int d^3\xi \Big\{ & \sqrt{-\gamma}\left[\gamma^{mn}\partial_m X^M \partial_n X^N g_{MN}(X) - 1\right] - \\ & - \frac{1}{3}\varepsilon^{mnp}\partial_m X^M \partial_n X^N \partial_p X^P b_{MNP}(X)\Big\}, \end{aligned}$$

as shown in [6].

We choose to work with the action (3.1), because it possesses the following advantages. First of all, it does not contain square root, thus avoiding the introduction of additional nonlinearities in the equations of motion. Besides, the equations of motion for the Lagrange multipliers λ^m generate the *independent* constraints only

$$\begin{aligned} G_{00} - 2\lambda^j G_{0j} + \lambda^i\lambda^j G_{ij} + \left(2\lambda^0 T_2\right)^2 \det\left(G_{ij}\right) &= 0\,, \\ G_{0j} - \lambda^i G_{ij} &= 0\,, \end{aligned} \tag{3.3}$$

where

$$G_{mn} = \partial_m X^M \partial_n X^N g_{MN}(X) \tag{3.4}$$

is the metric induced on the membrane worldvolume. Finally, this action gives a unified description for the tensile and tensionless membranes, so the limit $T_2 \to 0$ may be taken at any stage of our considerations.

Further on, we will use the gauge $\lambda^m = constants$, in which the equations of motion for X^M, following from (3.1), are given by ($\mathbf{G} \equiv \det\left(G_{ij}\right)$)

$$\begin{aligned} & g_{LN}\left[\left(\partial_0 - \lambda^i\partial_i\right)\left(\partial_0 - \lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_2\right)^2 \partial_i\left(\mathbf{G}G^{ij}\partial_j X^N\right)\right] + \\ & + \Gamma_{L,MN}\left[\left(\partial_0 - \lambda^i\partial_i\right)X^M\left(\partial_0 - \lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_2\right)^2 \mathbf{G}G^{ij}\partial_i X^M \partial_j X^N\right] = \\ & = 2\lambda^0 T_2 H_{LMNP}\partial_0 X^M \partial_1 X^N \partial_2 X^P\,, \end{aligned}$$

where

$$\Gamma_{L,MN} = g_{LK}\Gamma^K_{MN} = \frac{1}{2}\left(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}\right)$$

are the components of the symmetric connection compatible with the metric g_{MN} and H_{LMNP} is the field strength of the 3-form gauge potential b_{MNP}.

We will investigate the M2-brane dynamics in the framework of the following two types of embedding ($\Lambda^\mu_m = constants$)

$$X^\mu(\tau,\xi^i) = \Lambda^\mu_m \xi^m = \Lambda^\mu_0\tau + \Lambda^\mu_1\delta + \Lambda^\mu_2\sigma\,, \qquad X^a(\tau,\xi^i) = Y^a(\tau)\,, \tag{3.5}$$

[2]Namely this action has been used in [2]–[4].

and

$$X^\mu(\tau,\xi^i) = \Lambda_1^\mu \delta + \Lambda_2^\mu \sigma + Y^\mu(\tau), \qquad X^a(\tau,\xi^i) = Y^a(\tau)\,. \tag{3.6}$$

Here, the embedding coordinates $X^M(\tau,\xi^i)$ are divided into $X^M = (X^\mu, X^a)$, where $X^\mu(\tau,\xi^i)$ correspond to the space-time coordinates x^μ, on which the background fields do not depend

$$\partial_\mu g_{MN} = 0\,, \qquad \partial_\mu b_{MNP} = 0\,. \tag{3.7}$$

In other words, we suppose that there exist n_μ commuting Killing vectors $\partial/\partial x^\mu$, where n_μ is the number of the coordinates x^μ. The two ansatzes — (3.5) and (3.6), will be referred to as *linear gauges* and *general gauges*, in analogy with the name *static gauge* used for the embedding $X^m(\xi^n) = \xi^m$.

All formulas, necessary for our calculations in this section, are given in appendix.

3.1 Exact membrane solutions in linear gauges

Comparing the M2-brane embeddings (3.5) and (3.6), which we are going to explore, with the previously used ones (2.1), (2.3) and (2.4), one sees that only (2.4) is of the same type. Namely, it is particular case of (3.5), corresponding to $(X^\mu = X^{0,2,3}, X^a = X^1)$

$$\begin{aligned}
&\Lambda_0^0 = 1\,, \quad \Lambda_i^0 = 0\,, \quad \Lambda_1^2 = \sqrt{2}a\,, \quad \Lambda_0^2 = \Lambda_2^2 = 0\,, \quad \Lambda_2^3 = \sqrt{2}m\,, \quad \Lambda_0^3 = \Lambda_1^3 = 0\,,\\
&X^1(\tau,\delta,\sigma) \;=\; Y^1(\tau) = \rho(\tau)\,.
\end{aligned}$$

As far as classical membrane solution has been not given in [4], we begin with obtaining such solution, based on their ansatz (2.4). Let us first write down the two actions — (3.1) and (3.2), for the case under consideration. To this end, we need to compute the induced metric (3.4). It can be found by comparing (3.3) with (A.2), for example. Its nonzero components are

$$G_{00} = -(2l_pR)^2\left(\cosh^2\rho - \dot\rho^2\right), \quad G_{11} = (2l_pR)^2 a^2 \sinh^2\rho, \quad G_{22} = (2l_pR)^2 m^2 \sinh^2\rho\,.$$

Taking this into account, one receives

$$S^{NG} = -(2\pi)^2 T_2 (2l_pR)^3 am \int dt \sinh^2\rho \sqrt{\cosh^2\rho - \dot\rho^2}\,,$$

which reproduces the Nambu-Goto type action (2.6), used in [4], for

$$T_2 = \frac{1}{(2\pi)^2 l_p^3}\,.$$

Our action for this case is given by (A.1), and it reads

$$\begin{aligned}
S^{LG} \;=\; \frac{(2\pi l_pR)^2}{\lambda^0}\int dt\Big\{&\dot\rho^2 + \left[\left(\lambda^1 a\right)^2 + \left(\lambda^2 m\right)^2\right]\sinh^2\rho -\\
&- \cosh^2\rho - \left(2\lambda^0 T_2\right)^2 (2l_pR)^2 a^2 m^2 \sinh^4\rho\Big\}\,.
\end{aligned} \tag{3.8}$$

Since our membrane configuration is defined by (2.4), the relevant background is (2.5). It does not depend on $x^0 = t$, $x^2 = \psi_2$ and $x^3 = \psi_5$, i.e. we have three commuting Killing vectors $\partial/\partial t$, $\partial/\partial\psi_2$ and $\partial/\partial\psi_5$. Correspondingly, the lagrangian in (3.8) does not depend

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on $x^0 = t$, $x^2 = \psi_2$ and $x^3 = \psi_5$. Therefore, the conjugated momenta $P_0 = P_t$, $P_2 = P_{\psi_2}$ and $P_3 = P_{\psi_5}$ are conserved. From (A.5), one obtains the following explicit expressions for them

$$P_0 = -\frac{(2l_pR)^2}{2\lambda^0}\cosh^2\rho\,, \quad P_2 = -\sqrt{2}a\lambda^1\frac{(2l_pR)^2}{4\lambda^0}\sinh^2\rho, \quad P_3 = -\sqrt{2}m\lambda^2\frac{(2l_pR)^2}{4\lambda^0}\sinh^2\rho\,.$$

For compatibility of the membrane embedding with the constraints (A.3), the conditions (A.6) must be fulfilled. In the present case, they lead to $P_2 = P_3 = 0$. This means that we have to work in the worldvolume gauge $\lambda^i = 0$. Then

$$P_2 = P_3 \equiv 0\,, \qquad \Lambda_i^\mu P_\mu \equiv 0\,,$$

and the constraints (A.3) are also identically satisfied.

In linear gauges, there is another consistency condition - (A.7), which connect the membrane energy E with all the conserved momenta P_μ. For the embedding, we are considering, (A.7) just states that

$$E = -VP_0 = \text{const}\,.$$

Thus, in the framework of the ansatz (2.4), the only nontrivial conserved quantity is the M2-brane energy.

Finally, it remains to present the solution of the equations of motion (A.9) and of the constraint (A.8). Our background (2.5) depends on only one coordinate - $x^1 = \rho$. In this case, as explained in the appendix, the constraint (A.8) is first integral for the equation of motion for $\rho(\tau)$, and the general solution satisfying $\tau(\rho_0) = \tau_0$ is given by (A.19). For the case at hand, it reads

$$\tau(\rho) = \tau_0 + \int_{\rho_0}^{\rho} d\rho\left[\frac{\lambda^0 E}{(2\pi l_pR)^2} - \cosh^2\rho - \left(2\lambda^0 T_2\right)^2(2l_pR)^2a^2m^2\sinh^4\rho\right]^{-1/2}. \tag{3.9}$$

Let us now try to find a membrane solution based on more general embedding of the type (3.5), when the background seen by the M2-brane depends on two coordinates. To this end, we choose the following ansatz ($X^\mu = X^{0,3,4}, X^a = X^{1,2}$ in our notations)

$$\begin{aligned}
X^0(\tau,\delta,\sigma) &\equiv t(\tau,\delta,\sigma) = \tau,\\
X^1(\tau,\delta,\sigma) &= Y^1(\tau) = \rho(\tau)\,,\\
X^2(\tau,\delta,\sigma) &= Y^2(\tau) = \psi_1(\tau)\,,\\
X^3(\tau,\delta,\sigma) &\equiv \psi_2(\tau,\delta,\sigma) = \Lambda_1^3\delta + \Lambda_2^3\sigma\,,\\
X^4(\tau,\delta,\sigma) &\equiv \psi_5(\tau,\delta,\sigma) = \Lambda_1^4\delta + \Lambda_2^4\sigma\,.
\end{aligned} \tag{3.10}$$

Comparing with (2.2), one sees that the relevant background metric is

$$ds^2 = (2l_pR)^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(d\psi_1^2 + \cos^2\psi_1 d\psi_2^2 + \sin^2\psi_1 d\psi_5^2\right)\right]. \tag{3.11}$$

For the above membrane configuration, our action (A.1), in worldvolume gauge $\lambda^i = 0$, reads

$$\begin{aligned}
S^{LG} &= \frac{(2\pi l_p R)^2}{\lambda^0} \int dt \Big[\dot{\rho}^2 + \dot{\psi_1}^2 \sinh^2\rho - \cosh^2\rho \\
&\qquad\qquad - (2l_p R)^2 \left(2\lambda^0 T_2 \Delta\right)^2 \sinh^4\rho \sin^2\psi_1 \cos^2\psi_1 \Big] , \\
\Delta &= \Lambda_1^3 \Lambda_2^4 - \Lambda_1^4 \Lambda_2^3 .
\end{aligned} \tag{3.12}$$

The corresponding Nambu-Goto type action is

$$S^{NG} = -(2\pi)^2 (2l_p R)^3 T_2 \Delta \int dt \sinh^2\rho \sin\psi_1 \cos\psi_1 \sqrt{\cosh^2\rho - \dot{\rho}^2 - \dot{\psi_1}^2 \sinh^2\rho} . \tag{3.13}$$

According to (A.5) and (A.7), the conserved quantities are given by

$$E = -(2\pi)^2 P_0 = \frac{(4\pi l_p R)^2}{2\lambda^0} \cosh^2\rho , \qquad P_3 = P_4 = 0 .$$

The compatibility conditions (A.6), and therefore - the constraints (A.3), are identically satisfied. So, our next task is to solve the equations of motion (A.9) and the remaining constraint (A.8). As far as the background (3.11) is diagonal one, and depends on two coordinates, we can use the general expressions (A.25) and (A.26) for the first integrals of the equations (A.21), which also solve the constraint (A.22), if the conditions (A.23) and (A.24) are satisfied. Let us check if this is the case. The conditions (A.23) are fulfilled, because they take the form

$$\mathcal{A}_a^L \equiv 0, \qquad \frac{\partial}{\partial \psi_1} \sinh^2\rho \equiv 0 .$$

Consequently, it remains to satisfy the conditions (A.24). In the case at hand, they require, the right hand sides of (A.25) and (A.26) to depend only on $Y^1 = \rho$ and $Y^2 = \psi_1$ respectively. To see if this is true, let us write down the first integrals (A.25) and (A.26) explicitly

$$(2l_p R)^4 \dot{\rho}^2 = -\frac{D_2(\rho)}{\sinh^2\rho} \equiv F(\rho) \geq 0 , \tag{3.14}$$

$$\begin{aligned}
(2l_p R \sinh\rho)^4 \dot{\psi_1}^2 &= D_2(\rho) + (2l_p R \sinh\rho)^2 \mathcal{U}^L(\rho, \psi_1) \\
&= D_2(\rho) + (2l_p R \sinh\rho)^2 \times \\
&\times \left\{ \frac{\lambda^0 E}{\pi^2} - (2l_p R)^2 \left[\cosh^2\rho + (2l_p R)^2 \left(2\lambda^0 T_2 \Delta\right)^2 \sinh^4\rho \sin^2\psi_1 \cos^2\psi_1 \right] \right\} .
\end{aligned} \tag{3.15}$$

It is evident that the r.h.s. of the equation for $\dot{\rho}$ is a function only on ρ, while the r.h.s. of the equation for $\dot{\psi}_1$ is not a function only on ψ_1. Hence, the second of the conditions (A.24) remains unsatisfied in the general case. There exists, however, a particular case, when it can be fulfilled. As long as the four parameters $\Lambda_i^{3,4}$ in our ansatz (3.10) are still arbitrary, we can restrict them by the condition $\Delta = 0$, and choose the arbitrary function $D_2(\rho)$ as

$$D_2(\rho) = d^2 - (2l_pR\sinh\rho)^2\left[\frac{\lambda^0 E}{\pi^2} - (2l_pR)^2\cosh^2\rho\right] \leq 0, \qquad d^2 = \text{const}\,.$$

In this way, the r.h.s. of (3.15) became a constant and all integrability conditions (A.23), (A.24), are satisfied.[3]

The same result may be achieved by setting the membrane tension $T_2 = 0$, instead of $\Delta = 0$. In both cases, the solution of the equations (3.14) and (3.15) will correspond to a *null membrane*, because the determinant of the worldvolume metric is zero for this configuration. We note that such solution cannot be obtained by using the Nambu-Goto type action (3.13). It is identically zero in this case, while the action (3.12), which we are using, simplifies to

$$S_0^{LG} = \frac{(2\pi l_pR)^2}{\lambda^0}\int dt\left(\dot{\rho}^2 + \dot{\psi_1}^2\sinh^2\rho - \cosh^2\rho\right).$$

Let us turn to the more interesting case, when the M2-brane extends also on the S^4-part of the $AdS_7\times S^4$ background. To this aim, we choose the following embedding of type (3.5)[4]

$$\begin{aligned}
X^0(\tau,\delta,\sigma) &\equiv t(\tau,\delta,\sigma) = \Lambda_0^0\tau + \Lambda_1^0\delta + \Lambda_2^0\sigma\,,\\
X^1(\tau,\delta,\sigma) &= Y^1(\tau) = \rho(\tau)\,,\\
X^2(\tau,\delta,\sigma) &\equiv \psi_2(\tau,\delta,\sigma) = \Lambda_0^2\tau + \Lambda_1^2\delta + \Lambda_2^2\sigma\,,\\
X^3(\tau,\delta,\sigma) &\equiv \psi_5(\tau,\delta,\sigma) = \Lambda_0^3\tau + \Lambda_1^3\delta + \Lambda_2^3\sigma\,,\\
X^4(\tau,\delta,\sigma) &= Y^4(\tau) = \alpha(\tau)\,,\\
X^5(\tau,\delta,\sigma) &\equiv \theta(\tau,\delta,\sigma) = \Lambda_0^5\tau + \Lambda_1^5\delta + \Lambda_2^5\sigma\,.
\end{aligned} \tag{3.16}$$

The background seen by the membrane is ($\psi_1 = \pi/4$)

$$ds^2 = (2l_pR)^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \frac{1}{2}\sinh^2\rho\left(d\psi_2^2 + d\psi_5^2\right) + \frac{1}{4}\left(d\alpha^2 + \cos^2\alpha d\theta^2\right)\right], \tag{3.17}$$

and in our notations $X^\mu = X^{0,2,3,5}, X^a = X^{1,4}$.

For the ansatz (3.16) and in accordance with (A.7), the energy E is a linear combination of all conserved momenta P_μ

$$E = -(2\pi)^2\Lambda_0^\mu P_\mu = -\frac{(2\pi)^2}{2\lambda^0}\Lambda_0^\mu\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right)g_{\mu\nu}\,.$$

Actually, the compatibility conditions (A.6), (A.7), can be satisfied by expressing three of the free parameters through the others. If we choose to exclude Λ_1^5, Λ_2^5 and P_0, the

[3] How the equations (3.14) and (3.15) can be solved, we will explain on the example of the next case of membrane embedding, considered below.

[4] The M-theory background 3-form on S^4 is zero for this ansatz.

following equalities will hold

$$\Lambda_1^5 = \frac{1}{\Lambda_0^0 P_5}\left[\Lambda_1^0\left(E+\Lambda_0^5 P_5\right) - D_{02}P_2 - D_{03}P_3\right],$$
$$\Lambda_2^5 = \frac{1}{\Lambda_0^0 P_5}\left[\Lambda_2^0\left(E+\Lambda_0^5 P_5\right) - d_{02}P_2 - d_{03}P_3\right],$$
$$P_0 = -\frac{1}{\Lambda_0^0}\left(E+\Lambda_0^2 P_2+\Lambda_0^3 P_3+\Lambda_0^5 P_5\right).$$

Here

$$D_{02} = \Lambda_0^0\Lambda_1^2 - \Lambda_0^2\Lambda_1^0\,, \qquad D_{03} = \Lambda_0^0\Lambda_1^3 - \Lambda_0^3\Lambda_1^0\,,$$
$$d_{02} = \Lambda_0^0\Lambda_2^2 - \Lambda_0^2\Lambda_2^0\,, \qquad d_{03} = \Lambda_0^0\Lambda_2^3 - \Lambda_0^3\Lambda_2^0\,.$$

Our next task is to solve the equations of motion (A.21) and the constraint (A.22), where $\mathcal{A}_a = 0$, $\mathcal{G}_{aa} = (g_{11}, g_{44})$,

$$\begin{aligned}\mathcal{U} &= \frac{2\lambda^0 E}{(2\pi)^2} - \left(2\lambda^0 T_2\right)^2 \det\left(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}\right)\\ &= \frac{2\lambda^0 E}{(2\pi)^2} + (2l_pR)^4\left(\lambda^0 T_2\right)^2\Big\{\left[2\left(\Delta_{02}^2+\Delta_{03}^2\right)\cosh^2\rho - \Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho +\\ &\qquad + \frac{1}{2}\left[2\Delta_{05}^2\cosh^2\rho - \left(\Delta_{25}^2+\Delta_{35}^2\right)\sinh^2\rho\right]\cos^2\alpha\Big\},\end{aligned}$$

and $\Delta_{\mu\nu}^2 = \left(\Lambda_1^\mu\Lambda_2^\nu - \Lambda_1^\nu\Lambda_2^\mu\right)^2$. Now, contrary to the previously considered case, we have enough freedom to satisfy the integrability conditions (A.23) and (A.24), for arbitrary value of the membrane tension. To this end, we can choose

$$\begin{aligned}&\Delta_{25}^2+\Delta_{35}^2 = 2\Delta_{05}^2\,,\\ &D_4(\rho) = d - (l_pR)^2\Big\{\frac{2\lambda^0 E}{(2\pi)^2} + (l_pR)^4\left(4\lambda^0 T_2\right)^2 \times\\ &\qquad \times\left[2\left(\Delta_{02}^2+\Delta_{03}^2\right)\cosh^2\rho - \Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho\Big\} \leq 0\,,\end{aligned}$$

where d is arbitrary constant. After this choice, the first integrals (A.25) and (A.26) of the equations of motion for $\rho(\tau)$ and $\alpha(\tau)$ take the form

$$\left(g_{11}\dot\rho\right)^2 = -4D_4(\rho) \equiv \Phi_1(\rho) \geq 0\,, \tag{3.18}$$
$$\left(g_{44}\dot\alpha\right)^2 = d + (l_pR)^6\left(4\lambda^0 T_2\Delta_{05}\right)^2\cos^2\alpha \equiv \Phi_4(\alpha) \geq 0\,. \tag{3.19}$$

The general solutions of these equations are given by

$$\tau(\rho) = (2l_pR)^2\int\frac{d\rho}{\sqrt{\Phi_1(\rho)}}\,, \qquad \tau(\alpha) = (l_pR)^2\int\frac{d\alpha}{\sqrt{\Phi_4(\alpha)}}\,.$$

From (3.18) and (3.19), we can also find the orbit $\rho = \rho(\alpha)$:

$$4\int\frac{d\rho}{\sqrt{\Phi_1(\rho)}} = \int\frac{d\alpha}{\sqrt{\Phi_4(\alpha)}}\,.$$

3.2 Exact membrane solutions in general gauges

In this subsection, we will consider several M2-brane configurations in the framework of the ansatz (3.6), which corresponds to more general embedding than (3.5). Now, the membrane coordinates $X^\mu(\tau,\delta,\sigma)$ are allowed to vary non-linearly with the proper time τ.

To begin with, let us take the most general ansatz of type (3.6) for the background (2.5)

$$\begin{aligned} X^0(\tau,\delta,\sigma) \equiv t(\tau,\delta,\sigma) &= \Lambda_1^0\delta + \Lambda_2^0\sigma + Y^0(\tau)\,, \\ X^1(\tau,\delta,\sigma) &= Y^1(\tau) = \rho(\tau)\,, \\ X^2(\tau,\delta,\sigma) &\equiv \psi_2(\tau,\delta,\sigma) = \Lambda_1^2\delta + \Lambda_2^2\sigma + Y^2(\tau)\,, \\ X^3(\tau,\delta,\sigma) &\equiv \psi_5(\tau,\delta,\sigma) = \Lambda_1^3\delta + \Lambda_2^3\sigma + Y^3(\tau), \\ X^\mu &= X^{0,2,3}\,, \qquad X^a = X^1\,. \end{aligned} \tag{3.20}$$

The conserved momenta are given in (A.12), and for our case they read

$$P_\mu = \frac{g_{\mu\nu}}{2\lambda^0}\left(\dot{Y}^\nu - \lambda^j\Lambda_j^\nu\right). \tag{3.21}$$

In particular, the membrane energy is

$$E = -p_0 = -VP_0 = \frac{(4\pi l_p R)^2}{2\lambda^0}\cosh^2\rho\left(\dot{Y}^0 - \lambda^j\Lambda_j^0\right). \tag{3.22}$$

The compatibility conditions (A.13) are satisfied for

$$\Lambda_1^3 = \frac{1}{p_3}\left(\Lambda_1^0 E - \Lambda_1^2 p_2\right), \qquad \Lambda_2^3 = \frac{1}{p_3}\left(\Lambda_2^0 E - \Lambda_2^2 p_2\right),$$

and the following relations between the conserved quantities can be also derived from them

$$E = \frac{\Delta_{23}}{\Delta_{03}}p_2 = -\frac{\Delta_{23}}{\Delta_{02}}p_3\,.$$

The background (2.5) depends only on the ρ-coordinate. In this case, the general solution for the membrane coordinate $\rho(\tau)$ is given by (A.19), which for the case under consideration reduces to

$$\tau(\rho) = \tau_0 + \frac{1}{2\lambda^0}\int_{\rho_0}^{\rho}\frac{d\rho}{\sqrt{W(\rho)}}\,,$$

where

$$\begin{aligned} W(\rho) &= \frac{1}{(4\pi l_p R)^4}\left[\frac{E^2}{\cosh^2\rho} - \frac{2(p_2^2+p_3^2)}{\sinh^2\rho}\right] - \\ &- (l_p R)^2\left(\frac{T_2\Delta_{02}}{p_3}\right)^2\left[E^2\sinh^2\rho - 2(p_2^2+p_3^2)\cosh^2\rho\right]\sinh^2\rho\,. \end{aligned}$$

To see the difference between the membrane solutions, obtained in the framework of different type of embeddings, one can compare the above result with (3.9). Both solutions are for the same background (2.5).

Working with the ansatz (3.20), we have to write down also the solutions for the remaining M2-brane coordinates X^μ, given in the generic case in (A.20). These general solutions are as follows

$$X^0(\rho,\delta,\sigma) \equiv t(\rho,\delta,\sigma) = \Lambda_1^0\left[\lambda^1\tau(\rho)+\delta\right]+\Lambda_2^0\left[\lambda^2\tau(\rho)+\sigma\right]+\frac{E}{(4\pi l_p R)^2}\int_{\rho_0}^{\rho}\frac{d\rho}{\cosh^2\rho\sqrt{W(\rho)}},$$

$$X^2(\rho,\delta,\sigma) \equiv \psi_2(\rho,\delta,\sigma) = \Lambda_1^2\left[\lambda^1\tau(\rho)+\delta\right]+\Lambda_2^2\left[\lambda^2\tau(\rho)+\sigma\right]+\frac{2p_2}{(4\pi l_p R)^2}\int_{\rho_0}^{\rho}\frac{d\rho}{\sinh^2\rho\sqrt{W(\rho)}},$$

$$\begin{aligned} X^3(\rho,\delta,\sigma) &\equiv \psi_5(\rho,\delta,\sigma) \\ &= \frac{1}{p_3}\left\{\left(\Lambda_1^0 E-\Lambda_1^2 p_2\right)\left[\lambda^1\tau(\rho)+\delta\right]+\left(\Lambda_2^0 E-\Lambda_2^2 p_2\right)\left[\lambda^2\tau(\rho)+\sigma\right]\right\}+ \\ &\quad +\frac{2p_3}{(4\pi l_p R)^2}\int_{\rho_0}^{\rho}\frac{d\rho}{\sinh^2\rho\sqrt{W(\rho)}}. \end{aligned}$$

The next M2-brane configuration, we will consider, is based on the most general ansatz of type (3.6) for the background (3.17)

$$\begin{aligned} X^0(\tau,\delta,\sigma) &\equiv t(\tau,\delta,\sigma) = \Lambda_1^0\delta+\Lambda_2^0\sigma+Y^0(\tau), \\ X^1(\tau,\delta,\sigma) &= Y^1(\tau)=\rho(\tau), \\ X^2(\tau,\delta,\sigma) &\equiv \psi_2(\tau,\delta,\sigma) = \Lambda_1^2\delta+\Lambda_2^2\sigma+Y^2(\tau), \\ X^3(\tau,\delta,\sigma) &\equiv \psi_5(\tau,\delta,\sigma) = \Lambda_1^3\delta+\Lambda_2^3\sigma+Y^3(\tau), \\ X^4(\tau,\delta,\sigma) &= Y^4(\tau)=\alpha(\tau), \\ X^5(\tau,\delta,\sigma) &\equiv \theta(\tau,\delta,\sigma) = \Lambda_1^5\delta+\Lambda_2^5\sigma+Y^5(\tau), \\ X^\mu &= X^{0,2,3,5}, \qquad X^a = X^{1,4}. \end{aligned} \tag{3.23}$$

The expressions for the conserved momenta, and in particular for the membrane energy are the same as in (3.21) and (3.22). The compatibility conditions (A.13) are fulfilled identically, when

$$\Lambda_1^5=\frac{1}{p_5}\left(\Lambda_1^0 E-\Lambda_1^2 p_2-\Lambda_1^3 p_3\right), \qquad \Lambda_2^5=\frac{1}{p_5}\left(\Lambda_2^0 E-\Lambda_2^2 p_2-\Lambda_2^3 p_3\right).$$

As explained in appendix, we can now give three types of membrane solutions: when α is fixed, when ρ is fixed, and without fixing any of the coordinates α and ρ, on which the background (3.17) depends. In the first two cases, the formulas (A.19), (A.20) apply. In the last case, we can use (A.25) and (A.26), if we succeed to satisfy the integrability conditions (A.23), (A.24). In all these cases, the effective scalar potential $\mathcal{U}^A(\rho,\alpha)$ is [5]

$$\begin{aligned} \mathcal{U}^A(\rho,\alpha) &= \frac{(\lambda^0)^2}{(2\pi)^4(l_pR)^2}\left[\frac{E^2}{\cosh^2\rho}-\frac{2(p_2^2+p_3^2)}{\sinh^2\rho}-\frac{4p_5^2}{\cos^2\alpha}\right]+ \\ &\quad +(2l_pR)^4\left(\lambda^0 T_2\right)^2\left\{\left[2\left(\Delta_{02}^2+\Delta_{03}^2\right)\cosh^2\rho-\Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho+\right. \\ &\quad \left.+\frac{1}{2}\left[2\Delta_{05}^2\cosh^2\rho-\left(\Delta_{25}^2+\Delta_{35}^2\right)\sinh^2\rho\right]\cos^2\alpha\right\}. \end{aligned}$$

[5] The effective 1-form gauge potential $\mathcal{A}^A=0$.

For $\alpha = \alpha_0 =$ constant, one obtains the solution

$$\begin{aligned}
\tau\,(\rho) &= \tau_0 + 2l_pR\int_{\rho_0}^{\rho}\frac{d\rho}{\sqrt{\mathcal{U}^A(\rho,\alpha_0)}}\,,\\
X^0(\rho,\delta,\sigma) \equiv t(\rho,\delta,\sigma) &= \Lambda_1^0\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^0\left[\lambda^2\tau(\rho)+\sigma\right] +\\
&\quad + \frac{\lambda^0 E}{(2\pi)^2 l_pR}\int_{\rho_0}^{\rho}\frac{d\rho}{\cosh^2\rho\sqrt{\mathcal{U}^A(\rho,\alpha_0)}}\,,\\
X^2(\rho,\delta,\sigma) \equiv \psi_2(\rho,\delta,\sigma) &= \Lambda_1^2\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^2\left[\lambda^2\tau(\rho)+\sigma\right] +\\
&\quad + \frac{2\lambda^0 p_2}{(2\pi)^2 l_pR}\int_{\rho_0}^{\rho}\frac{d\rho}{\sinh^2\rho\sqrt{\mathcal{U}^A(\rho,\alpha_0)}}\,,\\
X^3(\rho,\delta,\sigma) \equiv \psi_5(\rho,\delta,\sigma) &= \Lambda_1^3\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^3\left[\lambda^2\tau(\rho)+\sigma\right] +\\
&\quad + \frac{2\lambda^0 p_3}{(2\pi)^2 l_pR}\int_{\rho_0}^{\rho}\frac{d\rho}{\sinh^2\rho\sqrt{\mathcal{U}^A(\rho,\alpha_0)}}\,,\\
X^5(\rho,\delta,\sigma) \equiv \theta(\rho,\delta,\sigma) &= \frac{1}{p_5}\Big\{\left(\Lambda_1^0E - \Lambda_1^2p_2 - \Lambda_1^3p_3\right)\left[\lambda^1\tau(\rho)+\delta\right] +\\
&\quad + \left(\Lambda_2^0E - \Lambda_2^2p_2 - \Lambda_2^3p_3\right)\left[\lambda^2\tau(\rho)+\sigma\right]\Big\}\\
&\quad + \frac{2\lambda^0 p_5}{(2\pi)^2(l_pR)^2\cos^2\alpha_0}\left[\tau(\rho)-\tau_0\right].
\end{aligned}$$

For $\rho = \rho_0 =$ constant, the M2-brane solution is

$$\begin{aligned}
\tau\,(\alpha) &= \tau_0 + l_pR\int_{\alpha_0}^{\alpha}\frac{d\alpha}{\sqrt{\mathcal{U}^A(\rho_0,\alpha)}}\,,\\
X^0(\alpha,\delta,\sigma) &\equiv t(\alpha,\delta,\sigma) = \Lambda_1^0\left[\lambda^1\tau(\alpha)+\delta\right] + \Lambda_2^0\left[\lambda^2\tau(\alpha)+\sigma\right] + \frac{2\lambda^0 E\left[\tau(\alpha)-\tau_0\right]}{(4\pi l_pR)^2\cosh^2\rho_0}\\
X^2(\alpha,\delta,\sigma) &\equiv \psi_2(\alpha,\delta,\sigma) = \Lambda_1^2\left[\lambda^1\tau(\alpha)+\delta\right] + \Lambda_2^2\left[\lambda^2\tau(\alpha)+\sigma\right] + \frac{4\lambda^0 p_2\left[\tau(\alpha)-\tau_0\right]}{(4\pi l_pR)^2\sinh^2\rho_0}\,,\\
X^3(\alpha,\delta,\sigma) &\equiv \psi_5(\alpha,\delta,\sigma) = \Lambda_1^3\left[\lambda^1\tau(\alpha)+\delta\right] + \Lambda_2^3\left[\lambda^2\tau(\alpha)+\sigma\right] + \frac{4\lambda^0 p_3\left[\tau(\alpha)-\tau_0\right]}{(4\pi l_pR)^2\sinh^2\rho_0}\,,\\
X^5(\alpha,\delta,\sigma) &\equiv \theta(\alpha,\delta,\sigma) = \frac{1}{p_5}\Big\{\left(\Lambda_1^0E - \Lambda_1^2p_2 - \Lambda_1^3p_3\right)\left[\lambda^1\tau(\alpha)+\delta\right] +\\
&\quad + \left(\Lambda_2^0E - \Lambda_2^2p_2 - \Lambda_2^3p_3\right)\left[\lambda^2\tau(\alpha)+\sigma\right]\Big\} +\\
&\quad + \frac{2\lambda^0 p_5}{(2\pi)^2 l_pR}\int_{\alpha_0}^{\alpha}\frac{d\alpha}{\cos^2\alpha\sqrt{\mathcal{U}^A(\rho_0,\alpha)}}\,.
\end{aligned}$$

When none of the coordinates ρ and α is kept fixed, the conditions (A.23), (A.24) will be fulfilled, if by using the arbitrariness of the parameters Λ_i^μ and of the function $D_4(\rho)$,

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we choose

$$
\begin{aligned}
\Delta_{25}^2 + \Delta_{35}^2 &= 2\Delta_{05}^2, \\
D_4(\rho) &= d - \frac{(\lambda^0)^2}{(2\pi)^4}\left[\frac{E^2}{\cosh^2\rho} - \frac{2(p_2^2+p_3^2)}{\sinh^2\rho}\right] - 16(l_pR)^6\left(\lambda^0 T_2\right)^2 \times \\
&\quad \times \left[2\left(\Delta_{02}^2+\Delta_{03}^2\right)\cosh^2\rho - \Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho \le 0\,,
\end{aligned}
$$

where d is arbitrary constant. After this choice is made, the first integrals (A.25) and (A.26) of the equations of motion for $\rho(\tau)$ and $\alpha(\tau)$ reduce to

$$
\begin{aligned}
(g_{11}\dot{\rho})^2 &= \frac{(2\lambda^0)^2}{(2\pi)^4}\left[\frac{E^2}{\cosh^2\rho} + \frac{2(p_2^2+p_3^2)}{\sinh^2\rho}\right] + (2l_pR)^6\left(\lambda^0 T_2\right)^2 \times \qquad (3.24)\\
&\quad \times \left[2\left(\Delta_{02}^2+\Delta_{03}^2\right)\cosh^2\rho - \Delta_{23}^2\sinh^2\rho\right]\sinh^2\rho - 4d \equiv F_1(\rho) \ge 0\,, \\
(g_{44}\dot{\alpha})^2 &= d + (l_pR)^6\left(4\lambda^0 T_2\Delta_{05}\right)^2\cos^2\alpha - \frac{(2\lambda^0 p_5)^2}{(2\pi)^4\cos^2\alpha} \equiv F_4(\alpha) \ge 0\,. \qquad (3.25)
\end{aligned}
$$

The general solutions of the above two equations are

$$
\tau(\rho) = (2l_pR)^2\int\frac{d\rho}{\sqrt{F_1(\rho)}}, \qquad \tau(\alpha) = (l_pR)^2\int\frac{d\alpha}{\sqrt{F_4(\alpha)}}\,.
$$

From (3.24) and (3.25), one can also find the orbit $\rho = \rho(\alpha)$:

$$
4\int\frac{d\rho}{\sqrt{F_1(\rho)}} = \int\frac{d\alpha}{\sqrt{F_4(\alpha)}}\,. \qquad (3.26)
$$

Now, we have to find the solutions for the remaining membrane coordinates X^μ. To this end, we will use (A.16), i.e. the conservation laws for p_μ, which in our case read

$$
\dot{Y}^\mu = \frac{2\lambda^0 p_\nu}{(2\pi)^2}\left(g^{-1}\right)^{\mu\nu}(\rho,\alpha) + \lambda^i\Lambda_i^\mu\,.
$$

Representing $\dot{Y}^\mu$ as

$$
\dot{Y}^\mu = \frac{\partial Y^\mu}{\partial\rho}\dot{\rho} + \frac{\partial Y^\mu}{\partial\alpha}\dot{\alpha}\,,
$$

and using (3.24) and (3.25), one obtains

$$
\frac{\sqrt{F_1(\rho)}}{(2l_pR)^2}\frac{\partial Y^\mu}{\partial\rho} + \frac{\sqrt{F_4(\alpha)}}{(l_pR)^2}\frac{\partial Y^\mu}{\partial\alpha} = \frac{2\lambda^0 p_\nu}{(2\pi)^2}\left(g^{-1}\right)^{\mu\nu}(\rho,\alpha) + \lambda^i\Lambda_i^\mu\,.
$$

This is a system of linear PDEs of first order, which general solution can be easily found. Its replacement in the ansatz (3.23), leads to the following explicit expressions for the M2-brane coordinates X^μ

$$
\begin{aligned}
X^0(\rho,\alpha;\delta,\sigma) \equiv t(\rho,\alpha;\delta,\sigma) &= \Lambda_1^0\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^0\left[\lambda^2\tau(\rho)+\sigma\right] + \\
&\quad + \frac{2\lambda^0 E}{(2\pi)^2}\int\frac{d\rho}{\cosh^2\rho\sqrt{F_1(\rho)}} + f^0[C(\rho,\alpha)]\,,
\end{aligned}
$$

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$$
\begin{aligned}
X^2(\rho,\alpha;\delta,\sigma) \equiv \psi_2(\rho,\alpha;\delta,\sigma) &= \Lambda_1^2\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^2\left[\lambda^2\tau(\rho)+\sigma\right] + \\
&\quad + \frac{4\lambda^0 p_2}{(2\pi)^2}\int \frac{d\rho}{\sinh^2\rho\sqrt{F_1(\rho)}} + f^2[C(\rho,\alpha)]\,, \\
X^3(\rho,\alpha;\delta,\sigma) \equiv \psi_5(\rho,\alpha;\delta,\sigma) &= \Lambda_1^3\left[\lambda^1\tau(\rho)+\delta\right] + \Lambda_2^3\left[\lambda^2\tau(\rho)+\sigma\right] + \\
&\quad + \frac{4\lambda^0 p_3}{(2\pi)^2}\int \frac{d\rho}{\sinh^2\rho\sqrt{F_1(\rho)}} + f^3[C(\rho,\alpha)]\,, \\
X^5(\rho,\alpha;\delta,\sigma) \equiv \theta(\rho,\alpha;\delta,\sigma) &= \frac{1}{p_5}\Big\{\left(\Lambda_1^0 E - \Lambda_1^2 p_2 - \Lambda_1^3 p_3\right)\left[\lambda^1\tau(\alpha)+\delta\right] + \\
&\quad + \left(\Lambda_2^0 E - \Lambda_2^2 p_2 - \Lambda_2^3 p_3\right)\left[\lambda^2\tau(\alpha)+\sigma\right]\Big\} + \\
&\quad + \frac{2\lambda^0 p_5}{(2\pi)^2}\int \frac{d\alpha}{\cos^2\alpha\sqrt{F_4(\alpha)}} + f^5[C(\rho,\alpha)]\,,
\end{aligned}
$$

where $f^\mu[C(\rho,\alpha)]$ are arbitrary functions of $C(\rho,\alpha)$. In turn, $C(\rho,\alpha)$ is the first integral of the equation (3.26).

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A. Generic formulas

Here, we describe the membrane dynamics and find the corresponding solutions of the equations of motion and constraints, in the framework of the two ansatzes — (3.5) and (3.6).[6] Initially, the background fields $g_{MN}(x)$ and $b_{MNP}(x)$ are restricted only by the conditions (3.7).

A.1 Membranes dynamics in linear gauges

In linear gauges, and under the conditions (3.7), the action (3.1) reduces to (the over-dot is used for $d/d\tau$)

$$
\begin{aligned}
S^{LG} &= \int d\tau L^{LG}(\tau), \qquad V = \int d^2\xi = \int d\delta d\sigma\,, \\
L^{LG}(\tau) &= \frac{V}{4\lambda^0}\Big\{g_{ab}\dot{Y}^a\dot{Y}^b + 2\Big[\left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)g_{\mu a} + 2\lambda^0 T_2 B_{a12}\Big]\dot{Y}^a + \\
&\quad + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right)g_{\mu\nu} - \left(2\lambda^0 T_2\right)^2\det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + 4\lambda^0 T_2\Lambda_0^\mu B_{\mu 12}\Big\}\,, \\
B_{M12} &\equiv b_{M\mu\nu}\Lambda_1^\mu\Lambda_2^\nu\,. \qquad \text{(A.1)}
\end{aligned}
$$

[6]We use part of the results obtained in [6], for the particular case $p = 2$ (2-brane).

The constraints derived from the lagrangian (A.1) are:

$$g_{ab}\dot{Y}^a\dot{Y}^b + 2\left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{\mu a}\dot{Y}^a + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) \times \tag{A.2}$$
$$\times \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + \left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) = 0\,,$$
$$\Lambda_i^\mu \left[g_{\mu a}\dot{Y}^a + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu}\right] = 0\,. \tag{A.3}$$

The lagrangian L^{LG} does not depend on τ explicitly, so the energy E is conserved:

$$g_{ab}\dot{Y}^a\dot{Y}^b - \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + \left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) - 4\lambda^0 T_2\Lambda_0^\mu B_{\mu 12} =$$
$$= \frac{4\lambda^0 E}{V} = \text{constant}.$$

With the help of the constraints (A.2) and (A.3), one can replace this equality by the following one

$$\Lambda_0^\mu \left[g_{\mu a}\dot{Y}^a + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} + 2\lambda^0 T_2 B_{\mu 12}\right] = -\frac{2\lambda^0 E}{V}\,. \tag{A.4}$$

In linear gauges, the momenta P_M take the form

$$2\lambda^0 P_M = g_{Ma}\dot{Y}^a + \left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{M\nu} + 2\lambda^0 T_2 B_{M12}\,. \tag{A.5}$$

The comparison of (A.5) with (A.3) and (A.4) gives

$$\Lambda_i^\mu P_\mu = \text{constants} = 0\,, \tag{A.6}$$
$$\Lambda_0^\mu P_\mu = -\frac{E}{V} = \text{constant}\,. \tag{A.7}$$

Therefore, in the linear gauges, the projections of the momenta P_μ onto Λ_n^μ are conserved. Moreover, as far as the lagrangian (A.1) does not depend on the coordinates X^μ, the corresponding conjugated momenta P_μ are also conserved.

The equalities (A.6) may be interpreted as solutions of the constraints (A.3), which restrict the number of the independent parameters in the theory.

Inserting (A.4) and (A.3) into (A.2), we obtain the *effective* constraint

$$g_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U}^L\,, \tag{A.8}$$

where the *effective* scalar potential is given by

$$\mathcal{U}^L = -\left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right)\left(\Lambda_0^\nu - \lambda^j\Lambda_j^\nu\right) g_{\mu\nu} +$$
$$+ 4\lambda^0\left(T_2\Lambda_0^\mu B_{\mu 12} + \frac{E}{V}\right).$$

In the gauge $\lambda^m = constants$, the equations of motion following from L^{LG} take the form:

$$g_{ab}\ddot{Y}^b + \Gamma_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a\mathcal{U}^L + 2\partial_{[a}\mathcal{A}_{b]}^L\dot{Y}^b\,, \tag{A.9}$$

where

$$\mathcal{A}_a^L = \left(\Lambda_0^\mu - \lambda^i\Lambda_i^\mu\right) g_{a\mu} + 2\lambda^0 T_2 B_{a12}\,,$$

is the *effective* 1-form gauge potential, generated by the non-diagonal components $g_{a\mu}$ of the background metric and by the components $b_{a\mu\nu}$ of the background 3-form gauge field.

A.2 Membranes dynamics in general gauges

We will use a superscript A to denote that the corresponding quantity is taken on the ansatz (3.6). It is understood that the conditions (3.7) are also fulfilled.

Now, the reduced lagrangian obtained from the action (3.1) is given by

$$
\begin{aligned}
L^A(\tau) = \frac{V}{4\lambda^0}[g_{MN}\dot{Y}^M\dot{Y}^N - 2\left(\lambda^i\Lambda_i^\mu g_{\mu N} - 2\lambda^0 T_2 B_{N12}\right)\dot{Y}^N + \\
+\ \lambda^i\Lambda_i^\mu\lambda^j\Lambda_j^\nu g_{\mu\nu} - \left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu})]\,.
\end{aligned}
$$

The constraints, derived from the above lagrangian, are:

$$g_{MN}\dot{Y}^M\dot{Y}^N - 2\lambda^i\Lambda_i^\mu g_{\mu N}\dot{Y}^N + \lambda^i\Lambda_i^\mu\lambda^j\Lambda_j^\nu g_{\mu\nu} + \left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) = 0\,, \tag{A.10}$$

$$\Lambda_i^\mu\left(g_{\mu N}\dot{Y}^N - \lambda^j\Lambda_j^\nu g_{\mu\nu}\right) = 0\,. \tag{A.11}$$

The corresponding momenta are ($P_M = p_M/V$)

$$2\lambda^0 P_M = g_{MN}\dot{Y}^N - \lambda^j\Lambda_j^\nu g_{M\nu} + 2\lambda^0 T_2 B_{M12}\,,$$

and part of them, P_μ, are conserved

$$g_{\mu N}\dot{Y}^N - \lambda^j\Lambda_j^\nu g_{\mu\nu} + 2\lambda^0 T_2 B_{\mu 12} = 2\lambda^0 P_\mu = \text{constants}\,, \tag{A.12}$$

because L^A does not depend on X^μ. From (A.11) and (A.12), the compatibility conditions follow

$$\Lambda_i^\mu P_\mu = 0\,. \tag{A.13}$$

We will regard on (A.13) as a solution of the constraints (A.11), which restricts the number of the independent parameters Λ_i^μ. That is why from now on, we will deal only with the constraint (A.10).

In the gauge $\lambda^m =$ constants, the equations of motion for Y^N, following from L^A, have the form

$$g_{LN}\ddot{Y}^N + \Gamma_{L,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_L\mathcal{U}^{in} + 2\partial_{[L}\mathcal{A}^{in}_{N]}\dot{Y}^N\,, \tag{A.14}$$

where

$$
\begin{aligned}
\mathcal{U}^{in} &= -\left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu\Lambda_j^\nu g_{\mu\nu}) + \lambda^i\Lambda_i^\mu\lambda^j\Lambda_j^\nu g_{\mu\nu}\,,\\
\mathcal{A}^{in}_N &= -\lambda^i\Lambda_i^\mu g_{N\mu} + 2\lambda^0 T_2 B_{N12}.
\end{aligned}
$$

Let us first consider this part of the equations of motion (A.14), which corresponds to $L = \lambda$. It is easy to check that they just express the fact that the momenta P_μ are conserved. Therefore, we have to deal only with the other part of the equations of motion, corresponding to $L = a$

$$g_{aN}\ddot{Y}^N + \Gamma_{a,MN}\dot{Y}^M\dot{Y}^N = \frac{1}{2}\partial_a\mathcal{U}^{in} + 2\partial_{[a}\mathcal{A}^{in}_{N]}\dot{Y}^N\,. \tag{A.15}$$

Our next task is to eliminate the variables $\dot{Y}^\mu$ from these equations and from the constraint (A.10). To this end, we will use the conservation laws (A.12) to express $\dot{Y}^\mu$ through $\dot{Y}^a$. The result is

$$\dot{Y}^\mu = \left(g^{-1}\right)^{\mu\nu} \left[2\lambda^0 (P_\nu - T_2 B_{\nu 12}) - g_{\nu a}\dot{Y}^a\right] + \lambda^i \Lambda_i^\mu \,. \tag{A.16}$$

By using (A.16), after some calculations, one rewrites the equations of motion (A.15) and the constraint (A.10) in the form

$$\begin{aligned} h_{ab}\ddot{Y}^b + \Gamma^{\mathbf{h}}_{a,bc}\dot{Y}^b\dot{Y}^c &= \frac{1}{2}\partial_a \mathcal{U}^A + 2\partial_{[a}\mathcal{A}^A_{b]}\dot{Y}^b \,, \\ h_{ab}\dot{Y}^a\dot{Y}^b &= \mathcal{U}^A \,, \end{aligned}$$

where a new, *effective metric* appeared

$$h_{ab} = g_{ab} - g_{a\mu}(g^{-1})^{\mu\nu} g_{\nu b} \,.$$

$\Gamma^{\mathbf{h}}_{a,bc}$ is the connection compatible with this metric

$$\Gamma^{\mathbf{h}}_{a,bc} = \frac{1}{2}\left(\partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc}\right) .$$

The new, *effective* scalar and gauge potentials are given by

$$\begin{aligned} \mathcal{U}^A &= -\left(2\lambda^0 T_2\right)^2 \det(\Lambda_i^\mu \Lambda_j^\nu g_{\mu\nu}) - (2\lambda^0)^2 \left(P_\mu - T_2 B_{\mu 12}\right) \left(g^{-1}\right)^{\mu\nu} \left(P_\nu - T_2 B_{\nu 12}\right) , \\ \mathcal{A}^A_a &= 2\lambda^0 \left[g_{a\mu}\left(g^{-1}\right)^{\mu\nu} \left(P_\nu - T_2 B_{\nu 12}\right) + T_2 B_{a12}\right] . \end{aligned}$$

A.3 Solutions of the equations of motion

The two cases of membrane dynamics considered so far, have one common feature. The dynamics of the corresponding reduced particle-like system is described by *effective* equations of motion and one *effective* constraint, which have the *same form*, independently of the ansatz used to reduce the membranes dynamics. Our aim here is to give their exact solutions. To be able to describe the two cases simultaneously, let us first introduce some general notations.

We will search for solutions of the following system of *nonlinear* differential equations

$$\mathcal{G}_{ab}\ddot{Y}^b + \Gamma^{\mathcal{G}}_{a,bc}\dot{Y}^b\dot{Y}^c = \frac{1}{2}\partial_a \mathcal{U} + 2\partial_{[a}\mathcal{A}_{b]}\dot{Y}^b \,, \tag{A.17}$$

$$\mathcal{G}_{ab}\dot{Y}^a\dot{Y}^b = \mathcal{U} \,, \tag{A.18}$$

where $\mathcal{G}_{ab}$, $\Gamma^{\mathcal{G}}_{a,bc}$, $\mathcal{U}$, and $\mathcal{A}_a$ can be as follows

$$\mathcal{G}_{ab} = (g_{ab}, h_{ab}) \,, \qquad \Gamma^{\mathcal{G}}_{a,bc} = \left(\Gamma_{a,bc}, \Gamma^{\mathbf{h}}_{a,bc}\right) , \qquad \mathcal{U} = \left(\mathcal{U}^L, \mathcal{U}^A\right) , \qquad \mathcal{A}_a = \left(\mathcal{A}^L_a, \mathcal{A}^A_a\right) ,$$

depending on the membrane embedding.

Let us start with the simplest case, when the background fields depend on only one coordinate $X^a = Y^a(\tau)$. In this case the solution of (A.17), compatible with (A.18), is just the constraint (A.18). In other words, (A.18) is first integral of the equation of motion for the coordinate Y^a. By integrating (A.18), one obtains the following exact membrane solution

$$\tau\left(X^{a}\right)=\tau_{0}+\int_{X_{0}^{a}}^{X^{a}}\left(\frac{\mathcal{U}}{\mathcal{G}_{aa}}\right)^{-1/2}dx\,, \tag{A.19}$$

where τ_0 and X_0^a are arbitrary constants.

When one works in the framework of the general ansatz (3.6), one has to also write down the solution for the remaining coordinates X^μ. It can be obtained as follows. One represents $\dot{Y}^\mu$ as

$$\dot{Y}^{\mu}=\frac{dY^{\mu}}{dY^{a}}\dot{Y}^{a}\,,$$

and use this and (A.18) in (A.16). The result is a system of ordinary differential equations of first order with separated variables, which integration is straightforward. Replacing the obtained solution for $Y^\mu(X^a)$ in the ansatz (3.6), one finally arrives at

$$\begin{aligned}X^{\mu}(X^{a},\xi^{i}) =\ & \Lambda_{i}^{\mu}\left[\lambda^{i}\tau(X^{a})+\xi^{i}\right]+\\ &+\int_{X_{0}^{a}}^{X^{a}}(g^{-1})^{\mu\nu}\left[2\lambda^{0}(P_{\nu}-T_{2}B_{\nu 12})\left(\frac{\mathcal{U}^{A}}{h_{aa}}\right)^{-1/2}-g_{\nu a}\right]dx\,. \end{aligned}\tag{A.20}$$

Let us turn to the more complicated case, when the background fields depend on more than one coordinate $X^a = Y^a(\tau)$. If the metric $\mathcal{G}_{ab}$ is a diagonal one, then the effective equations of motion (A.17) and the effective constraint (A.18) can be rewritten in the form

$$\begin{aligned}&\frac{d}{d\tau}\left(\mathcal{G}_{aa}\dot{Y}^{a}\right)^{2}-\dot{Y}^{a}\partial_{a}\left(\mathcal{G}_{aa}\mathcal{U}\right)+\\ &+\dot{Y}^{a}\sum_{b\neq a}\left[\partial_{a}\left(\frac{\mathcal{G}_{aa}}{\mathcal{G}_{bb}}\right)\left(\mathcal{G}_{bb}\dot{Y}^{b}\right)^{2}-4\partial_{[a}\mathcal{A}_{b]}\mathcal{G}_{aa}\dot{Y}^{b}\right]=0\,,\end{aligned}\tag{A.21}$$

$$\mathcal{G}_{aa}\left(\dot{Y}^{a}\right)^{2}+\sum_{b\neq a}\mathcal{G}_{bb}\left(\dot{Y}^{b}\right)^{2}=\mathcal{U}\,. \tag{A.22}$$

To find solutions of the above equations without choosing particular background, we can fix all coordinates X^a except one. Then the exact membrane solution of the equations of motion is given again by the same expression (A.19) for $\tau\left(X^a\right)$. In the case when one is using the general ansatz (3.6), the solution (A.20) still also hold.

To find solutions depending on more than one coordinate, we have to impose further restrictions on the background fields. We cannot give a prescription how to solve the problem in the general case. However, we can give an example of sufficient conditions, which are fulfilled in many cases, and which allow us to find the first integrals of the equations of motion (A.21), compatible with the effective constraint (A.22). If we denote one of the coordinates Y^a with Y^r and Y^α are the others, these conditions on the background can be written as

$$\mathcal{A}_{a}\equiv(\mathcal{A}_{r},\mathcal{A}_{\alpha})=(\mathcal{A}_{r},\partial_{\alpha}f)\,,\qquad \partial_{\alpha}\left(\frac{\mathcal{G}_{\alpha\alpha}}{\mathcal{G}_{aa}}\right)=0\,, \tag{A.23}$$

$$\partial_{\alpha}\left(\mathcal{G}_{rr}\dot{Y}^{r}\right)^{2}=0\,,\qquad \partial_{r}\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^{\alpha}\right)^{2}=0\,. \tag{A.24}$$

By using the restrictions given above, one obtains the following first integrals of the equations (A.21), which also solve the constraint (A.22)

$$\left(\mathcal{G}_{rr}\dot{Y}^r\right)^2 = \mathcal{G}_{rr}\left[(1-n_\alpha)\mathcal{U} - 2n_\alpha\left(\mathcal{A}_r - \partial_r f\right)\dot{Y}^r - \sum_\alpha \frac{D_\alpha\left(Y^{a\neq\alpha}\right)}{\mathcal{G}_{\alpha\alpha}}\right] = F_r\left(Y^r\right) \geq 0\,, \tag{A.25}$$

$$\left(\mathcal{G}_{\alpha\alpha}\dot{Y}^\alpha\right)^2 = D_\alpha\left(Y^{a\neq\alpha}\right) + \mathcal{G}_{\alpha\alpha}\left[\mathcal{U} + 2\left(\mathcal{A}_r - \partial_r f\right)\dot{Y}^r\right] = F_\alpha\left(Y^\beta\right) \geq 0\,, \tag{A.26}$$

where n_α is the number of the coordinates Y^α, and D_α, F_r, F_α are arbitrary functions of their arguments.

Further progress is possible, when working with particular background configurations, allowing for separation of the variables in (A.25) and (A.26).

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